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NUMBER AND FRACTIONS.

By J. K. ELLWOOD, A. M., Pittsburg, Pennsylvania.

A clear understanding of what *number* is and what gives rise to the number idea removes all difficulty from the grasping of the *fraction* idea.

Number does not inhere in objects, cannot be perceived by the senses; otherwise the mere presentation of 2, 3......n objects to the senses would give rise to the idea of number. There is in every sound mind a *measuring* instinct, which, in the nature of things, is just as essential to life and progress as is memory. Both the physical and ideal worlds are full of entities—vague wholes—which the mind must *measure* for the purpose of making them more definite. Measuring requires a "unit of measure." Naturally the first measurements made by a child are vague; as when he measures (counts) the chairs in a room, the marbles in his pocket, the fingers on his hand. His units of measure—chair, marble, finger—are indefinite, as are the results of his processes. A later stage involves *exact* measurements; i. e., an exactly defined unit of measure is used. A whole (of quantity), say a piece of cloth, is to be measured—made definite in value. A *yard* (exactly defined as 3 feet or 36 inches) is taken as the unit and applied (say) *ten* times. Then *ten* repetitions of the unit is the *number*. Considered by itself the *ten* is *pure number*, the result of a purely mental process; it expresses the *ratio* of the measured *quantity* to the measuring unit. Applied to the unit of measure, then ten expresses the numerical value of the measured quantity—10 yards of cloth. This ten yards, it is evident, is *quantity*, not number. It is what arithmetics erroneously call "concrete number." In this example the pure number indicates either of two things: (a) that the unit is taken *ten times*, or (b) that ten parts (units) are taken *one time*. It answers the question "how many?" Applied to the unit, it answers the question "how much?"

The number and unit of measure *together* give the absolute magnitude of the quantity; the number *alone* gives the relative value. Hence we may say that *number is the ratio of the quantity measured to the unit of measure.*

It is plain that *any* quantity may be used as a unit of measure. Measurement is more exact when this unit is itself made up of a definite number of equal parts—measured by some other unit, which may be called "primary" to distinguish it from the actual or direct unit of measure, which may be called "derived." Thus, if the unit of measure is three feet and it is taken ten times, we have the primary unit *one foot*, the derived unit *three feet*, and the number of derived units, *ten*. We have ten *threes*. To find the number of primary units we use multiplication, which gives thirty *ones*; the quantity is now more definite.

Again, in the quantity $5\times\$3$, the primary unit is \$1, the derived (direct, actual) unit \$3, five of which=15 primary units.

The derived unit is not necessarily a *multiple* of the primary unit; it may be one or more of its *equal parts*. Thus in $\$\frac{5}{2}$, the primary unit is, as above, \$1, while the derived unit is $\$\frac{1}{2}$, the number of them *five*. The fraction $\frac{5}{2}$ expresses the ratio of the measured quantity ($\$\frac{5}{2}$) to the primary unit (\$1). The numerator shows how many derived units make up the quantity, the denominator shows the relation between the derived and primary units. It is thus seen that the fraction involves no new idea. Its notation is more complete than that of the integer in that it defines the derived unit—makes *explicit* what is implied in the integral notation. This appears in the processes of finding the value of 5 hats (*a*) at \$3 each, (*b*) at $\$\frac{1}{2}$ each.

$$5\times\$3=5\times\overline{3\times\$1}=15\times\$1=\$15.$$
$$5\times\$\tfrac{1}{2}=5\times\overline{\tfrac{1}{2}\times\$1}=\tfrac{5}{2}\times\$1=\$\tfrac{5}{2}.$$

The denominator 2 shows the relation between the derived unit ($\$\frac{1}{2}$) and the primary unit (\$1). In \$15, however, there is nothing to show the relation between \$3 and \$1. (This is seen in $5\times\overline{3\times\$1}$). In no other respect does the fraction differ from the integer. Both 15 and $\frac{5}{2}$ express ratio to the primary unit \$1. The 15 shows the number of primary units, but not that of the derived units. The $\frac{5}{2}$ shows both; there are 5 derived units, $\frac{5}{2}$ primary units.

In view of these facts it appears that a correct definition of *number* includes that of *fraction*, which is simply a number whose notation gives a more complete statement of the mental processes by which number is constituted. For mathematical purposes *Newton's* definition cannot be much improved: "Number is the abstract ratio of one quantity to another quantity of the same kind." Ratio being a pure abstraction, the word "abstract" should be omitted. Euler says, "Number is the ratio of one quantity to another quantity taken as unit." Drs. McLellan and Dewey define number as, "The repetition of a certain magnitude used as the unit of measurement to equal or express the comparative value of a magnitude of the same kind."*

*In conclusion I wish to say that every live teacher should read "*The Psychology of Number*,"

It is clear that $\frac{1}{n}$ of any magnitude may be repeated as a unit just as well as $\frac{n}{n}$ or $\frac{3n}{n}$; it is equally plain that $\frac{m}{n}$ is as much an expression of ratio as is m. Hence each definition applies to fractions as well as integers.

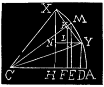
It is neither necessary nor advisable to divide ("break") single things (individuals, as apples) into parts in order to get fractions. In counting the eggs in a dozen (e. g.) the wee bairn is on the border of the fairyland of fractions, though he may not be conscious of it. At any stage of his counting the result is either integral or fractional. Five eggs is integral with respect to the *unit* (1 egg); it is fractional with respect to the *unity* or whole (dozen)—5 out of 12, 5 *twelfths*. Five *half-yards* is just as integral as 5 *yards*. The *ratio* in each is *five*. But in $\frac{5}{2}$ yards the ratio is $\frac{5}{2}$; the fractional idea is present, owing to the denominator, which defines the unit of measure.

SOME TRIGONOMETRIC RELATIONS PROVED GEOMETRICALLY.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

Most trigonometric formulæ may be proven geometrically in an elegant manner; and moreover, the relations between the trigonometric functions may be shown at a glance by means of the geometric figures. The results are all the more interesting, too, when proven also directly from first principles. For this reason the following exercises are offered.

For convenience, describe the arc AYX, and take the radius AC for the unit of measurement. Let the arc $AX=x$ and arc $AY=y$. Take M at the middle of XY, and draw lines as indicated.



Then $DY=\sin y$, $HX=\sin x$, $EM=\sin\frac{1}{2}(x+y)$, $KY=\sin\frac{1}{2}(x-y)$, $NX=\sin x-\sin y$, $NY=\cos y-\cos x$, $CE=\cos\frac{1}{2}(x+y)$, $CK=\cos\frac{1}{2}(x-y)$.

$$\text{Now, } HX+DY=2KF=2EM\frac{KF}{EM}=2EM\frac{CK}{CM}=2EM.CK.$$

That is, $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$ ....(1).

by Drs. McClellan and Dewey. It is interesting in matter, vigorous and aggressive in style, refreshing in its originality, and scholarly in its conception and execution. It is in the 33d volume of the International Education Series, published by D. Appleton & Co., New York.

Again, $CH+CD=2CF=2CE.\frac{CF}{CE}=2CE\frac{CK}{CM}=2CE.CK$,

$$\text{or } \cos x+\cos y=2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The triangles CEM and XNY are similar;

$$\text{hence } \frac{NX}{NY}=\frac{CE}{CM}, \text{ or } NX=2CE\frac{\frac{1}{2}XY}{CM}=2CE.KY,$$

$$\text{that is, } \sin x-\sin y=\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)\ldots\ldots\ldots\ldots(3).$$

Similarly, $\frac{NY}{XY}=\frac{EM}{CM}$, or $NY=2EM\frac{\frac{1}{2}XY}{CM}=2EM.KY$,

$$\text{or } \cos x-\cos y=-2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)\ldots\ldots\ldots\ldots\ldots(4).$$

Equation (1) can be made very useful in computing trigonometric tables, as the writer intends subsequently to show.

Now let $AM=x$ and $MY=MX=y$. Then $AY=x-y$ and $AX=x+y$. We have $(CM)^2-(CK)^2=(CY)^2-(CK)^2=(KY)^2$. But $\frac{CM}{ME}=\frac{CK}{KF}=\frac{KY}{LY}$.

Therefore $(ME)^2-(KF)^2=(LY)^2=(KY)^2-(KL)^2$,

or $(KF)^2-(KL)^2=(ME)^2-(KY)^2$,

or $(KF+KL)(KF-KL)=(ME)^2-(KY)^2$,

or $HX\times DY=(ME)^2-(KY)^2$.

That is, $\sin(x+y)\sin(x-y)=\sin^2x-\sin^2y=\cos^2y-\cos^2x\ldots\ldots\ldots\ldots\ldots\ldots(5).$

Again, $\frac{CM}{CE}=\frac{CK}{CF}=\frac{KY}{KL}$.

Therefore $(CE)^2-(CF)^2=(KL)^2=(KY)^2-(LY)^2$,

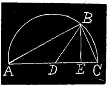
or $(CF)^2-(LY)^2=(CE)^2-(KY)^2$,

or $(CF-LY)(CF+LY)=CH\times CD=(CE)^2-(KY)^2$.

That is, $\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y=\cos^2y-\sin^2x\ldots\ldots\ldots\ldots\ldots\ldots(6).$

Let $DC=R$ the radius of a circle. Let the angle $CDB=2x$. Then $DAB=DBA=CBE=x$.

Then we have $\tan x=\frac{EC}{EB}$, also $\tan x=\frac{BE}{AE}$.



The product of these gives, $\tan^2x=\frac{CE}{AE}$, or $CE\times AE=(BE)^2$,

$$\text{or } \frac{EC}{AE}=\left(\frac{BE}{AE}\right)^2=\tan^2x.$$

Also, $\frac{EC}{BE}=\frac{\text{vers}2x}{\sin 2x}=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x}=\tan x$ [see above] ....(7).

Then $1+\tan^2 x=1+\frac{EC}{AE}=\frac{AC}{AE}=\frac{2R}{AE}$,

$$1-\tan^2 x=1-\frac{EC}{AE}=\frac{AE-EC}{AE}=\frac{AC-2EC}{AE}=\frac{2(R-EC)}{AE}.$$

$\cot 2x=\frac{DE}{BE}$ and $\csc 2x=\frac{R}{BE}$. From these values we at once have,

$$\frac{2\tan x}{1+\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2R}=\frac{BE}{R}=\sin 2x \;\ldots\ldots\ldots\ldots(8).$$

$$\frac{2\tan x}{1-\tan^2 x}=\frac{2BE}{AE}\cdot\frac{AE}{2(R-EC)}=\frac{BE}{R-EC}=\frac{BE}{DE}=\tan 2x \;\ldots\ldots\ldots\ldots(9).$$

$$\tan^2 x+2\cot 2x\tan x=\frac{EC}{AE}+\frac{2DE}{BE}\cdot\frac{BE}{AE}=\frac{EC+2DE}{AE}=\frac{AE}{AE}=1 \;\ldots\ldots\ldots\ldots(10).$$

$$2\csc 2x\tan x-\tan^2 x=\frac{2R}{BE}\cdot\frac{BE}{AE}-\frac{EC}{AE}=\frac{2R-EC}{AE}=\frac{AE}{AE}=1 \;\ldots\ldots\ldots\ldots(11).$$

$$\frac{1-\tan^2 x}{1+\tan^2 x}=\frac{2(R-EC)}{AE}\cdot\frac{AE}{2R}=\frac{R-CE}{R}=\frac{ED}{R}=\cos 2x \;\ldots\ldots\ldots\ldots(12).$$

$$\csc 2x-\cot 2x=\frac{R-ED}{BE}=\frac{EC}{BE}=\tan x=\frac{1-\cos 2x}{\sin 2x}=\frac{\sin 2x}{1+\cos 2x} \;\ldots\ldots\ldots\ldots(13).$$

$$\csc 2x+\cot 2x=\frac{R+ED}{BE}=\frac{AE}{BE}=\cot x=\frac{\sin 2x}{1-\cos 2x}=\frac{1+\cos 2x}{\sin 2x} \;\ldots\ldots\ldots\ldots(14).$$

$$\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x}=\frac{R+BE-ED}{R}\div\frac{R+BE+ED}{R}=\frac{EC+BE}{AE+BE}.$$

But $AE=\frac{(BE)^2}{EC}$; $\therefore\ \frac{CE+BE}{AE+BE}=\frac{EC+BE}{(BE)^2\div EC+BE}$

$$=\frac{EC(EC+BE)}{BE(EC+BE)}=\frac{EC}{BE}=\tan x \;\ldots\ldots\ldots\ldots(15).$$

Again, $\cos x=\dfrac{AE}{AB}$, also $\cos x=\dfrac{AB}{AC}$.

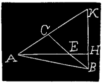
Twice the product of these gives $2\cos^2 x=\dfrac{2AE}{AC}=\dfrac{AE}{R}$.

Also $\cos 2x=\dfrac{DE}{R}$. $\quad 1+\cos 2x=\dfrac{DE+R}{R}=\dfrac{AE}{R}$. $\quad\therefore 1+\cos 2x=2\cos^2 x$......(16).

$\sin x=\dfrac{CB}{AC}=\dfrac{BC}{2R}$; also $\sin x=\dfrac{EC}{BC}$. Twice the product of these gives

$2\sin^2 x=\dfrac{EC}{R}$. $\quad 1-\cos 2x=\dfrac{R-ED}{R}=\dfrac{EC}{R}$. $\quad\therefore 1-\cos 2x=2\sin^2 x$..........(17).

To prove the "Tangent Proportion," let ABC be a plane triangle, the parts being represented as usual. Take $CE=CA$ and draw AEH. Draw BHK perpendicular to AH, to meet AC prolonged in K. Now considering the triangles ABC and ACE, the sum of the angles at A and E of the one is equal to the sum of the angles at A and B of the other. Hence $CAE+CEA=A+B$; and $CAE=CEA=BEH=\frac{1}{2}(A+B)$.



Also $BAE=A-\frac{1}{2}(A+B)=\frac{1}{2}(A-B)$. The angles at B and K of the triangle BCK are equal; for CBK is the complement of BEH or AEC, and BKC is the complement of the equal angle CAE. Hence $CK=CB=a$ and $AK=a+b$.

Now $\tan\frac{1}{2}(A-B)=\dfrac{BH}{AH}$ and $\tan\frac{1}{2}(A+B)=\dfrac{HK}{AH}$. $\quad\therefore \dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\dfrac{BH}{HK}$.

But $\dfrac{BH}{HK}=\dfrac{BE}{AK}=\dfrac{a-b}{a+b}$. $\quad\therefore \dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\dfrac{a-b}{a+b}$....................(1).

From the triangle ABE, $\dfrac{BE}{AB}=\dfrac{\sin BAE}{\sin AEC}$, or $\dfrac{a-b}{c}=\dfrac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}$..(2).

In the triangle AHK, $AH=AK\cos HAK=(a+b)\cos\frac{1}{2}(A+B)$.

In the triangle ABH, $AH=AB\cos BAH=c\cos\frac{1}{2}(A-B)$.

Equating, we have, $\dfrac{a+b}{c}=\dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$ ....(3).

Equation (3) divided by (2) also gives (1).

TWO PERPENDICULARS TO A TRANSVERSAL.

By JOHN N. LYLE, Ph. D., Bentonville, Arkansas.

Do two perpendiculars to a transversal intersect ?

Both Euclid and Lobatschewsky affirm that they do not. Euclid regards the two perpendiculars as equidistant, whilst Lobatschewsky considers them as diverging.

Experience confirms the view that the distance between the perpendiculars is a constant. As long as this is the case it is evident that intersection is impossible. If the perpendiculars do not approach each other within the range of observation and experience what would anology and induction indicate ? Would they not unmistakably favor the hypothesis that the perpendiculars do not intersect beyond the limits of observation and experience ? Our knowledge of *the here* and *the now*, if at all accurate, must assuredly count for something *elsewhere* and *tomorrow*.

But aside from conclusions based on purely empirical data and obtained by analogical and inductive processes the assumption that a straight line that has a beginning and an end is *infinite* involves contradiction and is therefore absurd. One end of each perpendicular is at the transversal. If these perpendiculars intersect each of them has two ends. But *two* ends is the distinctive characteristic of a finite straight line.

The further assumption that the intersection takes place at a hypothetical place called "infinity" does not remove the difficulty in the slightest. *Two* ends are still attributed to the *supposed* infinite line.

There is in reality a new difficulty and a very serious one, for the logical law of non-contradiction is violated.

The difficulty is not that the human mind by reason of its limited powers is unable to cognize an unlimited straight line and discover what will or will not take place "at infinity," but it is that the mind by reason of the logical law of non-contradiction can not cognize a line that is at the same time both unlimited and limited.

As a result of this brief investigation we find that there are insuperable difficulties, logical, geometrical, and philosophical, in the hypothesis that two perpendiculars to a transversal intersect at a supposed place called "infinity."

Notwithstanding these difficulties in the way of this hypothesis many analysts daily and habitually accept it. They do make the "assumption that parallel lines, extended to an infinite distance, do intersect."

Euclid flatly contradicts this hypothesis in his statement that "parallels never meet however far they may be produced." In favor of Euclid's statement there is nothing in logic, science or geometry known to man that conflicts with it. I understand Mr. Drummond's protest to extend not only to Euclid's assumption but also to the assumption that Euclid contradicts.

If the analysts "can not comprehend the infinite" why do they employ the symbol of the infinite so freely in their equations and decide without hesitation so many questions against the Alexandrian geometer? The analysts make large use of the symbol ∞ in their equations. Do they or do they not comprehend the meaning of the symbolism employed? If they find ∞ incomprehensible, can they not obtain all legitimate results by the aid of *finite* quantities alone?

DEVELOPMENT OF SINθ AND COSθ.

By J. M. BANDY, Trinity College, Trinity, North Carolina.

In discussing the power of the calculus with my own students in Trinity College, I, several years ago, sprung the question "why can the trigonometric functions, sine and cosine, be developed by series?"

The calculus very readily furnished the series; but it did not expose the exponential nature of the functions.

The fact that the value of the functions can be expressed by series forced me to the conclusion that the reason existed in the nature of the functions themselves, and, therefore, they should yield this result directly.

Before proceeding to obtain the series directly from the functions, it will be necessary to produce a series involving an exponential function. The object thereafter will be to trace the law which connects sine and cosine with this exponential function.

We will develop $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ which gives us a simple converging series. This series can be made to express an exponential function.

Denoting $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ by e; that is, as x increases indefinitely, the *limiting value* of this function $\left(1+\frac{1}{x}\right)^x\Big]_\infty$ is e.

$\therefore\ e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}$, etc.* From this we get

$$e^\theta=\left\{\left(1+\frac{1}{x}\right)^x\Big]_\infty\right\}^\theta=1+\theta+\frac{\theta^2}{1.2}+\frac{\theta^2}{1.2.3}+\text{etc.},\ \ldots\ldots\ldots\ldots (1),$$

$$e^{\frac{1}{x}}=\left\{\left(1+\frac{1}{x}\right)^x\Big]_\infty\right\}^{\frac{1}{x}}=1+\frac{1}{\infty},\ \ldots\ldots\ldots\ldots (2),$$

*This gives $e=2.71828$, the Naperian base.

and $\log\left(1+\frac{1}{\infty}\right)=\frac{1}{\infty}\log e$ . . . . (3).

To expose the principles which connect $\sin\theta$ and $\cos\theta$ with the above equations, and thus show that they can be expressed by series.

By geometry, $\cos^2\theta+\sin^2\theta=1$ . . . . (4).

The first member of (4) may be expressed thus: $\cos^2\theta-(-\sin^2\theta)=1$.

(4), therefore, becomes $\cos^2\theta-(-\sin^2\theta)=1$. . . . . (5).

Factoring first member of (5), we have,

$$(\cos\theta+\sin\theta\sqrt{-1})(\cos\theta-\sin\theta\sqrt{-1})=1 \quad \text{. . . . . (6).}$$

Taking log. of (6), we have $\log(\cos\theta+\sin\theta\sqrt{-1})+\log(\cos\theta-\sin\theta\sqrt{-1})=0$,

or $\log(\cos\theta+\sin\theta\sqrt{-1})=-\log(\cos\theta-\sin\theta\sqrt{-1})$. . . . . (7).

Denoting either member of (7) by y, we have,

$$\left.\begin{aligned}\log(\cos\theta+\sin\theta\sqrt{-1})&=y,\\ \text{and } \log(\cos\theta-\sin\theta\sqrt{-1})&=-y,\end{aligned}\right\} \quad \text{. (8).}$$

$\therefore\ \cos\theta+\sin\theta\sqrt{-1}=10^y$, ... (9), and $\cos\theta-\sin\theta\sqrt{-1}=10^{-y}$. . . . . (10).

Summing (9) and (10), $2\cos\theta=10^y+10^{-y}$. . . . . (11).

By trigonometry, $\cos^2\frac{1}{2}\theta=\frac{1}{2}(1+\cos\theta)=\frac{1}{4}(2+2\cos\theta)$

$=\frac{1}{4}(10^y+2+10^{-y})$, [from (11)]. . . . . (12),

and $-\sin^2\frac{1}{2}\theta=\frac{1}{2}(\cos\theta-1)=\frac{1}{4}(2\cos\theta-2)=\frac{1}{4}(10^y-2+10^{-y})$, [from (11)]. . . . (13).

Extracting square roots of (12) and (13),

$$\cos\tfrac{1}{2}\theta=10^{\frac{y}{2}}+10^{-\frac{y}{2}}, \quad \text{. . . . . (14),}$$

$$\text{and } \sin\tfrac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}-10^{-\frac{y}{2}} \quad \text{. (15).}$$

Adding (14) and (15), $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}$. (16).

Comparing (16) and (9), we see that θ may be changed into $\frac{1}{2}\theta$, provided that y is changed into $\frac{1}{2}y$. The same changes may, therefore, be made in (16): $\frac{1}{2}\theta$ may be changed into $\frac{1}{4}\theta$, if $\frac{1}{2}y$ is changed into $\frac{1}{4}y$. (16), therefore, becomes

$$\cos\tfrac{1}{4}\theta+\sin\tfrac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}} \quad \text{. (17).}$$

Repeating this change, we have, $\cos\frac{1}{8}\theta+\sin\frac{1}{8}\theta\sqrt{-1}=10^{\frac{y}{4}}$ ....(18).

Thus we see that θ may be divided by any power of 2, however great, provided y is divided by the same power.

Let, then, $m=2^n$ ....(19).

We then have, $\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1}=10^{\frac{y}{m}}$ ....(20).

Taking log of (20), we have, $\log(\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1})=\frac{y}{m}$ ...... (21).

But when n in (19) becomes infinite, m becomes infinite.

$\therefore$ $\cos\frac{1}{m}\theta$ in the limit equals 1, and $\sin\frac{1}{m}\theta\sqrt{-1}$ in the limit equals the arc. $\therefore$ (21) becomes $\log(1+\frac{\theta}{m}\sqrt{-1})=\frac{y}{m}$ ....(22).

But from (3), (22) becomes $\frac{\theta}{m}\sqrt{-1}\log e=\frac{y}{m}$, or $y=\theta\sqrt{-1}\log e$ .....(23).

Substituting this value of y in (8), $\log(\cos\theta+\sin\theta\sqrt{-1})=\theta\sqrt{-1}\log e$..(24),

and $\log(\cos\theta-\sin\theta\sqrt{-1})=-\theta\sqrt{-1}\log e$..........(25).

Whence $\cos\theta+\sin\theta\sqrt{-1}=e^{\theta\sqrt{-1}}$..........(26),

and $\cos\theta-\sin\theta\sqrt{-1}=e^{-\theta\sqrt{-1}}$ ....(27).

Adding (26) and (27), and dividing by 2, $\cos\theta=\frac{1}{2}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})$ ....(28),

by subtracting (27) from (26), and multiplying by $\sqrt{-1}$,

$\sin\theta=-\frac{1}{2}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})\sqrt{-1}$..........(29).

(28) and (29) enable us to develop $\cos\theta$ and $\sin\theta$ in a series arranged according to the powers of θ. Since $(\theta\sqrt{-1})^2=-\theta^2$, $(\theta\sqrt{-1})^3=-\theta^3\sqrt{-1}$, $(\theta\sqrt{-1})^4=\theta^4$, the substitution of $\theta\sqrt{-1}$ for θ in (1), gives

$$e^{\theta\sqrt{-1}}=1+\theta\sqrt{-1}-\frac{\theta^2}{1.2}-\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}+\frac{\theta^3\sqrt{-1}}{1.2.3.4.5} \text{ ....(30),}$$

$$\text{and } e^{-\theta\sqrt{-1}}=1-\theta\sqrt{-1}-\frac{\theta^2}{1.2}+\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}-\frac{\theta^3\sqrt{-1}}{1.2.3.4.5} \text{..........(31).}$$

Half the sum of (30) and (31) by (28) gives

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5.6} + \text{ etc.},$$

and half the difference of (30) and (31) by (29) gives

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^3}{1.2.3.4.5} - \text{ etc.}$$

The above are the required series. It is hoped that the law connecting $\cos\theta$ and $\sin\theta$ has been made plain.

(28) and (28) are Euler's results reached in a different way.

From (28) and (29) Demoivre's Theorem, which enables us to obtain the n roots of $y^n+1=0$ and $y^n-1=0$, is derived.

November 4, 1893.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6% ?

III. Solution by the PROPOSER.

Let x= equated time.

Now the amount of \$100 for $(x-2)$ years+the present worth of \$200 due $(4-x)$ years hence must=\$300.

$100+6(x-2)$=amount of \$100 for $(x-2)$ years at 6%.

$\frac{10000}{62-3x}$=present worth of \$200 due $(4-x)$ years hence at 6%.

$$\therefore\ 100+6(x-2)+\frac{10000}{62+3x}=300.$$

$\therefore\ x=3.31533+$years=3 years, 3 months, 24 days.

PROOF. \$107.89=amount of \$100 for 1.31533 years at 6%.

\$192.11=present worth of \$200 due 0.68467 year hence at 6%.

\$107.89+\$192.11=\$300.

QUERY: Will the answers prove as obtained to the solutions of this problem on page 238, Vol. III.?

64. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The following method of solution I have found to be infallible for all problems of Compound Proportion.

The first ratio (simple) has for its antecedent the *quantity to be found*, and for its consequent the corresponding similar quantity of the problem; hence $x:70$.

We now reason as follows: Work, time, etc., as the case may be, being equal, can a *longer* or *shorter* ditch be dug—

(1). By digging it 40 feet wide than by digging it 25 feet wide? Evidently shorter; hence 25:40.

(2). By digging it 3 feet deep than by digging it 4 feet deep? Longer; hence 4:3.

(3). With 15 men than with 27 men? Shorter; hence 15:27.

(4). In 16 days than in 10 days? Longer; hence 16:10.

(5). By working 9 hours a day than by working 7 hours? Longer; hence 9:7.

(6). With \$500 than with \$375? Longer; hence 500:375.

$$\text{Whence, } x:70::\left\{\begin{array}{r}25:40\\4:3\\15:27\\16:10\\9:7\\500:375\end{array}\right. \qquad \therefore x=88\tfrac{8}{9}.$$

Solved with same result by *P. S. BERG* and *EDWARD R. ROBBINS*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

There are two interpretations of the problem.

(1). The men are paid by the cubic foot; in this case the second lot should handle $\frac{500}{375}=\frac{4}{3}$ as much dirt as the first lot.

$$\therefore \tfrac{4}{3}\times\frac{70\times25\times4}{40\times3}=\tfrac{700}{9}=77\tfrac{7}{9} \text{ rods length of ditch.}$$

(2). Both ditches are dug by contract and the men are worked at their best all the time; in this case the amount received has nothing to do with the length of the ditch.

$$\therefore\left\{\begin{array}{c}27\\10\\7\end{array}\right\}:\left\{\begin{array}{c}70\\25\\4\end{array}\right\}=\left\{\begin{array}{c}15\\16\\9\end{array}\right\}:\left\{\begin{array}{c}x\\40\\3\end{array}\right\}.$$

$$\therefore x=\frac{70\times25\times4\times15\times16\times9}{27\times10\times7\times40\times3}=66\tfrac{2}{3}\text{ rods.}$$

[NOTE. $88\frac{8}{9}$ is obtained by multiplying $66\frac{2}{3}$ by $\frac{4}{3}$.]

Solved with same result as in (1) by *FREDERICK R. HONEY*.

[NOTE. There seems to be some disagreement among our contributors as to the correct solution of this problem. I, however, agree with Mr. Gruber, and have used his method of solution for several years. For a more detailed statement of this method the reader is referred to my *Mathematical Solution Book*. EDITOR.]

65. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Bought April 4, 1894, 250 yards of broadcloth at $\$5.37\frac{1}{2}$ per yard, less $12\frac{1}{2}$ and 10% discount for cash payment. Sold September 5, 1894, at 15, 10, and 5% on *quoted price*, the cloth; and in settlement received a 90-day note which I had discounted at $5\frac{1}{2}$%, October 19, 1894, by the First National Bank of Baltimore, Maryland. Reckoning 6% interest on the *money invested* in the cloth, what is the profit made?

I. Solution by P. S. BERG, Larimore, North Dakota, and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

$1.00-.12\frac{1}{2}=.87\frac{1}{2}$, $1.00-.10=.90$. $\therefore$ $1.00\times.87\frac{1}{2}\times.90=78\frac{3}{4}\%$.

$\$5.37\frac{1}{2}\times.78\frac{3}{4}\times250=\$1058.203125=$ cost.

From April 4th to September 5th is 5 months, 1 day, at 6%, \$1. amounts to $\$1.025\frac{1}{6}$. $\$1058.203125\times1.025\frac{1}{6}=\$1084.8336$.

$1.00+.15=1.15$, $1.00+.10=1.10$, $1.00+.05=105\%$.

$1.15\times1.10\times1.05=132.825\%$.

$\$5.37\frac{1}{2}\times1.32825\times250=\$1784.8359375$.

From October 19th to December 8th, 50 days, at $5\frac{1}{2}\%$, $\$1.=.007\frac{23}{36}$.

$\$1.00-\$.007\frac{23}{36}=\$.992\frac{13}{36}$.

$\$1784\ 8359375\times.992\frac{13}{36}=\$1771.2017$.

$\$1771.2017-\$1084.8336=\$686.3681$ profit.

PROBLEMS.

69. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

Every man in a certain group belongs to at least one of these classes: Methodists, Democrats, Farmers. In the group there are 10 Methodists, 12 Democrats, 13 Farmers; 3 men who are Methodists and Democrats, 4 who are Democrats and Farmers, 5 who are Methodists and Farmers. Finally, there are 2 men who are at the same time Methodists, Democrats and Farmers. Required the number of men in the group.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years; when can I pay him \$100 to settle the account equitably, money being worth 6%?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the loci of the foci of variable ellipses passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having for diameters the semi-latas rectum of the fixed ellipse.

Solution by the PROPOSER.

If the given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(1),

the equation to the required ellipse is of the form $\frac{x^2}{a_1{}^2}+\frac{(y-n)^2}{b_1{}^2}=1$ ....(2).

This passing through $(ae, 0)$, we have $\frac{a^2e^2}{a_1{}^2}+\frac{n^2}{b_1{}^2}=1$ ....(3).

The directrix of (2) is $x=\frac{a_1}{e_1}$...(4), e_1 being the eccentricity of (2), and $x=a$..(5) is the tangent to the given ellipse at the extremity of its major axis. Then

$\frac{a_1}{e_1}=a$............(6), or $a_1=ae_1$............(7), $a_1e_1=ae_1{}^2$...........(8).

Let (x', y') be the coordinates of the right hand focus of (2) in any one of its positions; then $a_1e_1=x'$..... ......(9), $n=y'$..... ......(10), and by (8) and (9), $e_1{}^2=\frac{x'}{a}$, $1-e_1{}^2=\frac{a-x'}{a}$ ....(11).

Also by (7), $a_1{}^2=a^2e_1{}^2=ax'$. ....(12);

$\therefore$ $b_1{}^2=a_1{}^2(1-e_1{}^2)=x'(a-x')$ ....(13), and (3) becomes

$\frac{a^2e^2}{ax'}+\frac{y'^2}{x'(a-x')}=1$ ....(14).

Reducing, $x'^2+y'^2-a(1+e^2)x'=-a^2e^2$............(15), a circle whose center is on the axis of x, passing through $(a, 0)$, and having diameter $\frac{b^2}{a}$.

Also solved by *G. B. M. ZERR.*

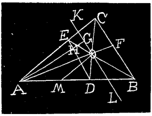
61. Proposed by **WILLIAM E. HEAL**, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O and meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the escribed circles.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Using trilinear coordinates, equation to CD is $\alpha-\beta=0$; to BE, $\alpha-\gamma=0$.

$$\therefore \left(\frac{2\triangle}{a+b},\ \frac{2\triangle}{a+b},\ 0\right),\ \left(\frac{2\triangle}{a+c},\ 0,\ \frac{2\triangle}{a+c}\right),$$



are the coordinates of D, E.

$\therefore$ $\beta+\gamma-\alpha=0$, is the equation to DE. The distance from O, (r, r, r) from this line is,

$$p_1=\frac{r}{\sqrt{3abc+2\cos C-2\cos A+2\cos B}}$$

$$=\frac{r}{\sqrt{\dfrac{3abc+a^2c+b^2c-c^3-ab^2-ac^2-b^3+a^2b+bc^2+a^3}{abc}}}$$

$$=\frac{r}{\sqrt{\dfrac{abc+(a+b+c)(a+b-c)(a-b+c)}{abc}}}=\frac{r}{\sqrt{\dfrac{abc+8s(s-b)(s-c)}{abc}}}$$

$$=\frac{r}{\sqrt{\dfrac{abc(s-a)+8\triangle^2}{abc(s-a)}}}=\frac{r}{\sqrt{\dfrac{\dfrac{abc}{4\triangle}+\dfrac{2\triangle}{s-a}}{\dfrac{abc}{4\triangle}}}}=\frac{r}{\sqrt{\dfrac{R+2r_1}{R}}}=\frac{rR}{\sqrt{R^2+2Rr_1}}=\frac{rR}{d_1}.$$

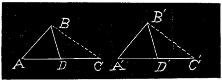
Similarly $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$.

62. Proposed by **F. P. MATZ**, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median of one of them* equal, each to each.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland, and CHAS. C. CROSS, Laytonsville, Maryland.

Let $AB=A'B'$, $AC=A'C'$, $BD=B'D'$. $\triangle ABD=\triangle A'B'D'$, because all the sides are equal, each to each.



Then $\triangle BDC=\triangle B'D'C'$, having two sides and included angle of one= two sides and included angle of the other.

$\therefore$ $\triangle ABC=\triangle A'B'C'$.

Also solved by *EDWARD R. ROBBINS, M. A. GRUBER*, and *G. B. M. ZERR.*

63. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Mississippi.**

A rectangular hyperbola cannot be cut from a right circular cone *if the angle at its vertex is less than a right angle.*

Solution by the PROPOSER.

Let the base and the axis of the cone coincide with the xy-plane and the z-axis respectively. Then if c denote the altitude of the cone and ϕ the angle which any one of its elements makes with the base, its equation is

$$(x^2+y^2)\tan^2\phi=(z-c)^2.$$

The equation of a plane through the y-axis and inclined at an angle θ to the xy-plane is

$$z=x\tan\theta.$$

The projection on the xy-plane of the intersection of the two surfaces is

$$(x^2+y^2)\tan^2\phi=(x\tan\theta-c)^2=x^2\tan^2\theta-2cx\tan\theta+c^2.$$

This becomes, when referred to rectangular axes in the plane of the section, the origin and y-axis being unchanged, $(x^2\cos^2\theta+y^2)\tan^2\phi=x^2\sin^2\theta-2cx\sin\theta+c^2$, or $x^2(\cos^2\theta\tan^2\phi-\sin^2\theta)+y^2\tan^2\phi+2cx\sin\theta-c^2=0$, which represents a rectangular hyperbola if $\tan^2\phi+\cos^2\theta\tan^2\phi-\sin^2\theta=0$. From this equation,

$$\sin^2\theta=\frac{2\tan^2\phi}{\tan^2\phi+1}=2\sin^2\phi, \text{ and } \sin\theta=\pm\sqrt{2}\sin\phi.$$

Since $\sin\phi$ cannot be greater than $\frac{1}{\sqrt{2}}$, ϕ cannot exceed 45°. Hence the angle at the vertex of the cone cannot be less than 90°.

Other solutions of this problem will appear in next issue.

PROBLEMS.

67. **Proposed by F. M. PRIEST, St. Louis, Mo.**

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago; Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with $A_1, A_2, A_3, \ldots A_{\frac{n}{2}-1}$, where n is supposed to be odd.

Prove that $OA_1 - OA_2 + OA_3 - OA_4 + \ldots \pm OA_{\frac{n}{2}-1}$, every other chord being affected with the minus sign.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A vertical slit is made in the middle of the side of a rectangular box containing water. What is the time required to empty the box?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let a, b, h = length, width, and depth of box, c = width of slit, m = coefficient of contraction, z = distance of surface of water from bottom of box, x = distance of any elemental area of slit from bottom of box.

$\therefore$ The quantity discharged through the slit in an element of time is

$$Q = [mc\sqrt{2g}\int_0^z \sqrt{z-x}\,dx]dt = \tfrac{2}{3}mc\sqrt{2g}\,z^{\frac{3}{2}}\,dt = abdz.$$

$$\therefore\ t = \frac{3ab}{2mc\sqrt{2g}}\int_{h'}^{h}\frac{dz}{z^{\frac{3}{2}}} = \frac{3ab(\sqrt{h}-\sqrt{h'})}{mc\sqrt{2ghh'}},\ \text{for depth } (h-h').$$

When $h'=0$, t = infinity or it is impossible to absolutely empty the box.

II. Solution by the PROPOSER.

Let x = distance from base of box to any point in the vertical slit below surface of water.

Let y = distance from base of box to surface of water.

The velocity of discharge for point $x = \sqrt{2g(y-x)}$.

$\therefore\ dF=k\sqrt{2g(y-x)}dx$, where k=width of slit, and F=flow of water.

Whence $F=k\sqrt{2g}\int_0^y \sqrt{y-x}\,dx=\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

Call V the volume of water in the box at any instant.

Then $\frac{dV}{dt}=\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$. But $V=aby$, where a and b are the dimensions of base of box. $\therefore\ \frac{abdy}{dt}=\frac{2k\sqrt{2g}}{3}y^{\frac{3}{2}}$.

From which $t=\frac{3ab}{2k}\int_n^m y^{-\frac{3}{2}}dy=\frac{ab}{k\sqrt{2g}}\left[\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{m}}\right]$, m and n being the depths of water at beginning and end of time of discharge.

If $n=0$, or the box is emptied, $t=\infty$.

If $m=\infty$, $t=\frac{ab}{k\sqrt{2gn}}$; or the time to empty a box of infinite depth to a finite depth is finite.

37. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.**

A thin board, of which the elements are given, is balanced at the center but inclined at an angle. A sphere of known dimensions is put directly above the point of suspension and liberated. Find the motion of the system. That is, find (a) the time until the sphere leaves the board, (b) the ultimate angular velocity of the board.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Take the horizontal line through the point making the greatest angle with the plane in its initial position as the axis of x, and the axis of y vertically downward through the same point. Let R and T be the normal and tangential reactions of the plane and sphere at any time t from the commencement of motion, θ and ϕ the angles of rotation of the sphere and of the plane, m, k, a the mass, radius of gyration, and radius of the sphere, and r=the distance the sphere has moved on the plane, and x and y the coordinates of the center of the sphere.

Resolving horizontally and vertically, and taking moments about the center of the sphere,

$$m\frac{d^2x}{dt^2}=R\sin\phi-T\cos\phi\,\ldots\ldots\ldots(1),\quad m\frac{d^2y}{dt^2}=mg-R\cos\phi-T\sin\phi\,\ldots\ldots\ldots(2).$$

$$mk^2\frac{d^2\theta}{dt^2}=Ta\,\ldots\ldots\ldots(3).\quad \text{We also have } \theta=\frac{r}{a}+\phi\,\ldots\ldots\ldots(4),$$

$$x=r\cos\phi+a\sin\phi\,\ldots\ldots\ldots(5),\text{ and } y=r\sin\phi-a\cos\phi\,\ldots\ldots\ldots(6).$$

Eliminating T from (1) and (2), $\sin\phi\frac{d^2x}{dt^2}-\cos\phi\frac{d^2y}{dt^2}=\frac{R}{m}-g\cos\phi$ ... (7).

Eliminating T and R from (1), (2), and (3),

$$\cos\phi\frac{d^2x}{dt^2}+\sin\phi\frac{d^2y}{dt^2}=g\sin\phi-\frac{k^2}{a}\frac{d^2\theta}{dt^2} \quad\text{.........(8).}$$

From (4), $\frac{d^2\theta}{dt^2}=\frac{1}{a}\frac{d^2r}{dt^2}+\frac{d^2\phi}{dt^2}$ ....(9).

From (5) and (6), $$\frac{d^2x}{dt^2}=\cos\phi\frac{d^2r}{dt^2}-2\sin\phi\frac{dr}{dt}\frac{d\phi}{dt}-r\cos\phi\frac{d\phi^2}{dt^2}$$

$$-r\sin\phi\frac{d^2\phi}{dt^2}-a\sin\phi\frac{d\phi^2}{dt^2}+a\cos\phi\frac{d^2\phi}{dt^2} \quad\text{.........(10),}$$

$$\frac{d^2y}{dt^2}=\sin\phi\frac{d^2r}{dt^2}+2\cos\phi\frac{dr}{dt}\frac{d\phi}{dt}-r\sin\phi\frac{d\phi^2}{dt^2}$$

$$+r\cos\phi\frac{d^2\phi}{dt^2}+a\cos\phi\frac{d\phi^2}{dt^2}+a\sin\phi\frac{d^2\phi}{dt^2} \quad\text{.........(11).}$$

Eliminating $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2\phi}{dt^2}$ from (7) and (8),

$$2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}+a\frac{d\phi^2}{dt^2}=g\cos\phi-\frac{R}{m} \quad\text{........(12),}$$

$$\frac{a^2+k^2}{a^2}\frac{d^2r}{dt^2}-r\frac{d\phi^2}{dt^2}+\frac{a^2+k^2}{a^2}\frac{d^2\phi}{dt^2}=g\sin\phi \quad\text{.........(13).}$$

(12) and (13) seem to indicate that one more condition at least should be given.

PROBLEMS.

46. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket
Ninety times as high as the moon." *Mother Goose.*

Neglecting the resistance of the air, how long did it take the old lady to go up?

47. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

What is the focus of the convex surface of a plano-convex lens, index μ, which will converge parallel monochromatic rays to a given focus, the rays entering the lens on the plane side?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Solve the equation $x^3+y^2=a^2$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Put $y=\dfrac{x(x-n^2)}{2n}$. Then we readily obtain $x^3+\left\{\dfrac{x(x-n^2)}{2n}\right\}^2=\left\{\dfrac{x(x+n^2)}{2n}\right\}^2$, which is a general formula for finding *the sum of a cube aud a square equal to a square*, x and n representing any values. We have also the general condition, derived from the formula, $nx+y=a$. By taking $n=1$, and putting $x=$, consecutively, the natural numbers beginning with unity, we obtain a series of equations in which the consecutive values both of y and a form *the series of integral numbers the sum of any two consecutive terms of which is the square of their difference.* [Problem 43, page 370, Vol. II.]

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $y=mx$, then $x^3+m^2x^2=a^2$. $\therefore$ $x+m^2=a^2/x^2=b^2$, $\therefore$ $b^2-m^2=x$, where b and m can be any integers $b>m$. We append some values.

b	m	x	y	a
1	0	1	0	1
2	1	3	3	6
3	2	5	10	15
4	3	7	21	28
5	4	9	36	45
&c.	&c.	&c.	&c.	&c.

III. Solution by M. C. STEVENS, M. A., Department of Mathematics, Purdue University, Layfayette, Indiana.

If x be any integer and $y=\dfrac{x(x-1)}{2}$, then $x^3+y^2=\dfrac{x^4+2x^3+x^2}{4}=a^2$.

$\therefore$ $a=\dfrac{x(x+1)}{2}$. If $x=1$, then $a=1$. If $x=2$, then $a=3$, and so on.
$y=0$ $\qquad$ $y=1$

IV. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

We write $x^3=a^2-y^2$. From the well known form

$mn=\left(\dfrac{m+n}{2}\right)^2-\left(\dfrac{m-n}{2}\right)^2$, if $x^3=mn$, the problem is answered.

Let m and n be 4 and 2; or 27 and 1; or 9 and 3; etc.; then $2^3+1^2=3^2$; $3^3+13^2=14^2$; $3^3+3^2=6^2$; etc.

V. Solution by H. C. WILKES, Skull Run, West Virginia.

$x^3=(a+y)(a-y)$. Let $a+y=x^2$ and $a-y=x$, then $x^2+x=2a$, and $x=\frac{1}{2}\pm\sqrt{2a+\frac{1}{4}}$. Let a be any triangular number, and from the above formula, integral values for x, a, and y can be found.

VI. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let $x=ky$. Then $x^3+y^2=a^2$ becomes $y^2\{k^3y+1\}=a^2$. This will be a square if $y=k^3+2$. $\therefore$ $y=k^3+2$, and $x=k(k^3+2)$ will be a solution, where k is any integer. If $k=1$, $y=3$, $x=3$ and $x^3+y^2=36$. If $k=2$, $y=10$, $x=20$, and $x^3+y^2=8100$, etc., etc.

VII. Solution by J. H. DRUMMOND, LL. D., Portland, Maine.

(A). If the problem is to be taken literally, $y=\sqrt{a^2-x^3}$ in which x may any number whose third power $<$ than a^2. But this does not give exact results.

(B). If it means that $x^3+y^2=\square$, let $x=my$ and we have $m^3y+1=\square=$ (say) b^2 and $y=(b^2-1)/m^3$ and $x=(b^2-1)/m^2$; but then $a=b(b^2-1)/m^3$, in which m and b may be any numbers greater than unity, but the value of a depends on x and y.

(C). By transposing $x^3=a^2-y^2$; take $x=a-y$, then $x^2=a+y$, and $a^2-2ay+y^2=a+y$, and $y=(2a+1\pm\sqrt{8a+1})/2$. As y must be less than a to make x positive, the sign of the radical term must be negative. It is readily seen that $a=n(n+1)/2$ makes $8a+1$ a square, and by reducing we get $y=n(n-1)/2$ and $x=n$, in which n may be any number.

(D). If the question means to find exact values of x and y for *any* value of a, I cannot solve it.

46. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In $x^2+x\sqrt{xy}=a$......(1) and $y^2+y\sqrt{xy}=b$......(2) find such values of a find b as will make x and y integral; give a general solution.

I. Solution by the PROPOSER.

Take $y=m^2x$, and by combining the two equations and reducing we have, $\frac{b}{a}(m+1)=m^3(m+1)$ and consequently $m^3=\frac{b}{a}$.

From (1) we have $x=\pm\sqrt{\frac{a}{m+1}}$. Take $a=c^2$ and $m+1=d^2$ and substituting, we have $x=c/d$. To make this value integral, take $c=de$; then $x=e$, and $y=m^2x=e(d^2-1)^2$. But $a=c^2$, and $c=dx=de$. $\therefore$ $a=d^2e^2$; but $b=am^3=d^2e^2(d^2-1)^3$, in which a may be any whole number>1, and e any whole number.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In order that $\sqrt{xy}$ be integral and rational, we put $x=rm^2$ and $y=rn^2$, r, m, and n being any integers. Whence we readily find that when $a=r^2m^3(m+n)$ and $b=r^2n^3(m+n)$, x and y are integral.

Now put $r=1$, $m=3$, and $n=2$, and we obtain $x^2+x\sqrt{xy}=135$ and $y^2+y\sqrt{xy}=40$; whence $x=9$ and $y=4$.

Put $r=2$, $m=2$, and $n=1$; then $x^2+x\sqrt{xy}=96$ and $y^2+y\sqrt{xy}=12$; whence $x=8$, and $y=2$.

III. Solution by A. H. BELL, Box 184, Hillsboro, Illinois.

The only condition to fill is to make $xy=\square$. Take $x=4$, $y=1$, and $a=24$, $b=3$, etc., etc.

IV. Solution by H. C. WILKES, Skull Run, West Virginia.

Let $m^2=x$, $n^2=y$. Then $m^3(m+n)=a$; $n^3(m+n)=b$. ∴ To make x and y integral, a and b must have a common factor $(m+n)$. The remaining factors will be m^3 and n^3. Let $a=448$, $b=189$; then $x=16$, $y=9$. $7(64)$, $m=4$; $7(27)$, $n=3$.

V. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $P=x^{\frac12}$, $Q=y^{\frac12}$. Then $P^4+P^3Q=a$......(1). $Q^4+Q^3P=b$.....(2).

$(1)\div(2)$, $P/Q=\sqrt[3]{a/b}$. $\therefore P^2=x=\pm\dfrac{a^{\frac23}}{\sqrt{a^{\frac13}+b^{\frac13}}}$, $Q^2=y=\pm\dfrac{b^{\frac23}}{\sqrt{a^{\frac13}+b^{\frac13}}}$.

Let $a=\{\frac12(m^2+n^2)\}^3$, $b=\{\frac12(m^2-n^2)\}^3$.

$\therefore x=\pm\dfrac{(m^2+n^2)^2}{4m}$, $y=\pm\dfrac{(m^2-n^2)^2}{4m}$.

Let $m=pn$. $\therefore x=\pm\dfrac{n^3(p^2+1)^2}{4p}$, $y=\pm\dfrac{n^3(p^2-1)^2}{4p}$.

Let $n=2p$. $\therefore x=\pm2p^2(p^2+1)^2$, $y=\pm2p^2(p^2-1)^2$.
$\therefore a=\{2p^2(p^2+1)\}^3$, $b=\{2p^2(p^2-1)\}^3$.

VI. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland

Let $y=m^2x$. Then $x^2(1+m)=a$, and $x^2m^3(1+m)=b$.

$\therefore m=\dfrac{b^{\frac13}}{a^{\frac13}}$. $x=\dfrac{a^{\frac23}}{\sqrt{a^{\frac13}+b^{\frac13}}}$; $y=\dfrac{b^{\frac23}}{\sqrt{a^{\frac13}+b^{\frac13}}}$.

Let $a=p^3$; $b=q^3$. Then $x=\dfrac{p^2}{\sqrt{p+q}}$; $y=\dfrac{q^2}{\sqrt{p+q}}$.

Let $p=2rs$; $q=r^2+s^2$. Then $x=\dfrac{4r^2s^2}{r+s}$; $y=\dfrac{(r^2+s^2)^2}{r+s}$.

Let $r=k+l$; $s=k-l$. Then $x=\dfrac{2(k^2-l^2)^2}{k}$; $y=\dfrac{2(k^2+l^2)^2}{k}$.

Let $l=\alpha k$. Then $x=2k^3(1-\alpha^2)^2$; $y=2k^3(1+\alpha^2)^2$.

Now $a=p^3=8r^3s^3=8(k^2-l^2)^3=8k^6(1-\alpha^2)^3$, and $b=q^3=(r^2+s^2)^3=8(k^2+l^2)^3=8k^6(1+\alpha^2)^3$, where α and k are integers.

PROBLEMS.

53. **Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.**

Given $x^2-114\frac{1}{4}y^2=\mp3$ to find the least values of x and y in integers.

54. **Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.**

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

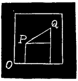
SOLUTIONS OF PROBLEMS.

35. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Find the chance that the distance of two points within a square shall not exceed a side of the square. [From *Byerly's Integral Calculus.*]

I. Solution by ALWYN C. SMITH, The University of Colorado, Boulder, Colorado.

a is one side of the square; P and Q the two points; (x, y) the point P with O for origin; and r and ϕ the polar coordinates of Q, with P as origin. Then the favorable cases are

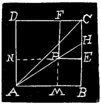


$$4\int_0^{\frac{1}{2}\pi}\int_0^a\int_0^{a-r\sin\phi}\int_0^{a-r\cos\phi}dxdyrdrd\phi=a^4(\pi-\tfrac{13}{6}).$$

All the cases $=a^2.a^2=a^4$. Therefore, $p=\pi-\frac{13}{6}$.

II. Solution by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Let $a=$a side of the square $ABCD$, and join A with any point P within the given square. Then as AP represents the distance and direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.



Let $AP=x$, $AH=x'$, and $\angle PAB=\theta$.

When $x'=a\sec\theta$, $PF=a-x\sin\theta$, $PE=a-x\cos\theta$.

$\therefore$ Area $PECF=(a-x\sin\theta)(a-x\cos\theta)$.

Hence the required chance is

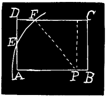
$$p_1=\frac{\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}{\int_0^{\frac{1}{4}\pi}\int_0^{x'} (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx}$$

$$=\frac{8}{a^4}\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)x\,d\theta\,dx$$

$$=\tfrac{4}{3}\int_0^{\frac{1}{4}\pi}(6-4\sin\theta-4\cos\theta+3\sin\theta\cos\theta)d\theta=\pi-\tfrac{13}{6}.$$

III. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Take a rectangle $ABCD$, with sides $AB=b$, and $BC=a$, such that a is not greater than b; and consider the chance that the proposed distance shall exceed b. Let N be the number of favorable cases; then if a be increased infinitesimally, dN will be the number of new cases introduced by placing each point in turn on the differential slice along b while the other one traverses the mixtilinear area DEF.



That is, taking AP equal to x,

$$dN=4\left[\int_{\sqrt{b^2-a^2}}^b \left(ax-\frac{a}{2}\sqrt{b^2-a^2}-\frac{x}{2}\sqrt{b^2-x^2}-\frac{b^2}{2}\sin^{-1}\frac{x}{b}+\frac{b^2}{2}\cos^{-1}\frac{a}{b}\right)dx\right]da$$

$$=2\left[2ab-ab\sqrt{b^2-a^2}-\frac{a^3}{3}-\frac{\pi b^3}{2}+b^3\cos^{-1}\frac{a}{b}\right]da\text{; and,}$$

$$N=2\left[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^4}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}\right]+C.$$

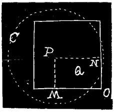
Since $N=0$ when $a=0$, $C=\frac{4b^4}{3}$; and,

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^2}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}+\frac{2b^4}{3}].$$

If now $a=b$, $N=b^4(\frac{19}{6}-\pi)$; and the whole number of cases $=b^4$. Hence the chance that the proposed distance shall exceed b is $\frac{19}{6}-\pi$; therefore, the chance that it will not exceed b is $\pi-\frac{13}{6}$.

IV. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Let a be one side of the square, and O the origin. With center O and radius a describe a quadrant. Let P any point within the square (x, y) be one point, and Q be the other point. With center P and radius a describe the circle C. Now Q may be anywhere within the area common to this circle and the square. The favorable cases may then be found by confining Q within the rectangle xy while P traverses the entire square, and then taking four times the result. Hence,



$$p=\frac{4}{a^4}\left\{\int_0^a\int_0^{\sqrt{a^2-y^2}}xydxdy+\frac{1}{2}\int_0^a\int_{\sqrt{a^2-y^2}}^a[x\sqrt{a^2-x^2}\right.$$

$$\left.+y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{x}{a}-a^2\cos^{-1}\frac{y}{a}]dxdy\right\}=\pi-\frac{13}{6}.$$

V. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

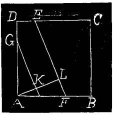
This problem affords a splendid test of the correctness of the general value for any convex area as demonstrated in problem 25, page 281, September-October MONTHLY.

Let AK, $AL=p$, $\angle LAB=\theta$, EF, $GH=C$.

For EF, $C=a\sec\theta$; the limits of p are $a\sin\theta$ to $a\cos\theta$.

For GH, $C=p\sec\theta\operatorname{cosec}\theta$; the limits of p are $a\sin\theta\cos\theta$ to $a\sin\theta$. The limits of θ are 0 to $\frac{1}{4}\pi$.

From problem 25,



$$\triangle=\frac{1}{3A^2}\int\int(C^3-3a^2C+2a^3)d\theta dp.$$

$$\therefore\ \triangle=\frac{8}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta\cos\theta}^{a\sin\theta}(p^3\sec^3\operatorname{cosec}^3\theta-3a^2p\sec\theta\operatorname{cosec}\theta+2a^3)d\theta dp$$

$$+\frac{4}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta}^{a\cos\theta}(a^3\sec^3\theta-3a^3\sec\theta+2a^3)d\theta dp.$$

$$\triangle=\tfrac{2}{3}\int_0^{\frac{1}{4}\pi}(\tan\theta\sec^2\theta-3\sin\theta\cos\theta-6\tan\theta+8\sin\theta)d\theta$$

$$+\tfrac{4}{3}\int_0^{\frac{1}{4}\pi}(\sec^2\theta-\tan\theta\sec^2\theta-3+3\tan\theta+2\cos\theta-2\sin\theta)d\theta.$$

$\therefore\ \triangle=\frac{19}{6}-\pi.\quad p=1-\triangle=\pi-\frac{13}{6}=$required chance.

PROBLEMS.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

45. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

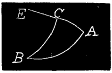
SOLUTIONS OF PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42°30'$ north $=\lambda$, at what angle with the horizon will the sun rise, its declination $=22°$ north $=\delta$?

I. Solution by the PROPOSER.

Let BA be a portion of the equator, $CA=\delta$, a portion of a meridian passing through the sun at C when rising, and describing a small-circle arc CE, parallel with BA, and let BC be a portion of the horizon. Then the angles ECA, and BAC, each $=90°$, because meridians cut the equator and circles of declination at right angles. Now $CBA=90°-\lambda$, then $\sin BCA=\sin\lambda\sec\delta=\cos BCE$. $\therefore BCE=43°\ 13'\ 37''=$required angle.



II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $x^2+z^2=R^2$..............(1) be the equation to the horison. Then, $x\cos\lambda+y\sin\lambda=R\sin\delta$..............(2) is the equation to the plane of the sun's path. (1) and (2) intersect in the points

$$x=\frac{R\sin\delta}{\cos\lambda},\quad y=0,\quad z=\pm\frac{R}{\cos\lambda}\sqrt{\cos^2\lambda-\sin^2\delta}.$$

The equation for the tangent plane for z positive is,

$$x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda.$$

$\therefore$ We must find the angle ϕ between the two lines in space,

$$\left.\begin{array}{r}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ x\cos\lambda+y\sin\lambda=R\sin\delta\end{array}\right\}\text{..................(3)},$$

$$\left.\begin{array}{r}x\sin\delta+z\sqrt{\cos^2\lambda-\sin^2\delta}=R\cos\lambda\\ y=0\end{array}\right\}\text{..................(4)}.$$

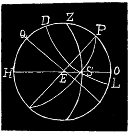
Let $s=\dfrac{\sin\delta}{\sqrt{\cos^2\lambda-\sin^2\delta}}$, $t=\cot\lambda$.

Then $\cos\phi=\dfrac{1+s^2}{\sqrt{(1+s^2)(1+s^2+t^2)}}=\sqrt{\dfrac{1+s^2}{1+s^2+t^2}}$. $\therefore\cos\phi=\dfrac{\sin\lambda}{\cos\delta}$.

$\therefore\ \phi=43°\ 13'\ 37''$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let HO represent the horizon, Z the zenith of the place of observation, EQ the equator, P the north pole, DL the diurnal circle of the sun, and S the position of the sun at rising. In the quadrantal spherical triangle ZPS, we have $ZS=90°$, $ZP=90°-\lambda$, $PS=90°-\delta$.



We find $\cos ZSP=\sin\lambda/\cos\delta$, but $\angle ZSP$ is equal to the angle which a tangent at S of the circle DSL makes with the horizon.

$\therefore\ \cos x=\dfrac{\sin\lambda}{\cos\delta}$, denoting by x the required angle. For the given concrete values we find $x=43°\ 14'$.

[NOTE.—Prof. H. C. Whittaker obtained as the numerical result for the required angle 70 degrees, 21 minutes, 56 seconds, and Prof. E. W. Morrell obtained 72 degrees, 9 minutes, 4.2 seconds. The result given in the published solutions seems to us to be the correct one.]

39. Proposed by SETH PRATT, C. E., Assyria, Michigan.

The pendulum of a clock which gains 6 seconds in 1 hour and 13 minute, makes 6000 vibrations in 1 hour and $9\frac{1}{2}$ minutes. What is the length of the pendulum? And what length should it have to keep true time?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Regarding 1 hour, 13 minutes and 1 hour, $9\frac{1}{2}$ minutes as registered by a clock keeping correct time, $g=32.16$, $\pi=3.1416$, $t=\pi\sqrt{(l/g)}$. Then 1 hour, $9\frac{1}{2}$ minutes$=4170$ seconds.

$$\therefore\ t=\tfrac{4170}{6000}=\tfrac{139}{200}=\pi\sqrt{\frac{l}{g}}.\quad \therefore\ l=\frac{(139)^2g}{(200\pi)^2}=1.57393 \text{ ft.}=18.88716 \text{ inches.}$$

1 hour, 13 minutes$=4380$ seconds.

$$\frac{4380\times200}{139}=\text{number of vibrations in 1 hour, 13 seconds.}$$

$$\therefore\ \frac{4380\times200}{139}=4386 \text{ seconds.}$$

$$\therefore\ t'=\frac{4386\times139}{4880\times200}=\frac{731\times139}{730\times200}=\pi\sqrt{\frac{l'}{g}}.$$

$$\therefore\ l'=\frac{(731\times139)^2g}{(730\times200\pi)^2}=1.578243 \text{ feet.}$$

$\therefore\ l'=18.93892$ inches$=$length to keep true time.

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

1 hour and $9\frac{1}{2}$ minutes$=4170$ seconds. 4170 seconds$\div6000=.695$ seconds, the time of one vibration. From Mechanics $l=t^2g/\pi^2$, whence $l=18.886$ inches, the length of the pendulum. Again, 1 hour and 13 minutes$=4380$ seconds. $4380\div.695=876/.139=$number of vibrations in 1 hour and 13 minutes. As the pendulum gains 6 seconds in that time, $6\div(876/.139)=.834/876=.0095$, the time in seconds gained in one vibration.

$\therefore$ .695 seconds$+$.0095 seconds$=$.69595 seconds, the time of vibrations of pendulum to keep correct time. Hence by substitutions in the above formula $l=18.9379$ inches, the length of pendulum to keep true time.

[NOTE.—The results sent in with the problem by the Proposer were, 18.89835 f inches, and for true time .036036 r inches longer. Prof. P. S. Berg in his solution obtained for length of pendulum 18.837975 inches, and 22.393 inches as the length to keep true time. EDITOR.]

PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Maryland.

Give a general proof that the centre of gravity, or centroid, determines that point from which the sum of the distances to all other points of a given area is the minimum.

50. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Describe and compute the actual path traversed by the moon in July and August, 1896, taking into account the motion of the earth around the sun.

51. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to a city, and bought 156 horses and 177 mules for \$23631; he then traveled farther due north to A, and bought at same price 468 horses and 235 mules for \$52245; he then traveled from A due west 130 miles to B, and bought 120 cows; he then traveled due north to C, and bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 goats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 as much for goats as sheep; at D he turned and traveled in a straight line to the city, a distance equal to the sum of the entire distance he traveled due north from his home H; he sold all his stock at a profit of 20%. How far did he travel from his home H the entire trip around and back to the city? What was the cost of each head of stock, and what was the entire gain?

52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Pennsylvania.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes 1-4 of its circumference at the base.

NOTES.

NOTE ON ARTICLE IN AUGUST-SEPTEMBER NUMBER, VOL. III.

BY WARREN HOLDEN.

Referring to the demonstration on page 207 (current volume) without disputing the conclusion, allow me to submit the following considerations:

In Algebra, when zero is a factor in any term, the product is zero. Accordingly $0\times\infty=0$. In the course of the demonstration appears the expression $\frac{0\times 1}{0}=\frac{0}{0}$, or the denominators being equal, $0\times 1=0$. Would this result affect the conclusion in any way?

NOTE ON ELIMINATION.

BY J. C. CORBIN, PINE BLUFF, ARKANSAS.

The operation of elimination by addition and subtraction may often be shortened by the process and rule given below:

I. $5x+7y=43$.

$11x+9y=69$.

To eliminate y. $(9\times5-7\times11)x=9\times43-7\times69$. $\therefore x=3$.

To eliminate x. $(11\times7-5\times9)y=11\times43-5\times69$. $\therefore y=4$.

II. $21x+20y=165$.

$77x-30y=295$.

To eliminate x. $(3\times21+2\times77)y=3\times165+2\times295$. $\therefore y=5$.

To eliminate y. $(11\times20+3\times30)x=11\times165-3\times295$.

This is, substantially, the Determinant method ; but it is derived from the ordinary algebraic process by omitting all unessential work. The rule is : The difference (sum) of the products containing x (y) is equal to the difference (sum) of the numerical products.

EDITORIALS.

A few complete sets of Vol. I. and Vol. II. are still left. We will send Vol. I. to any address in the United States for \$2., and Vol. II. for \$2.50. Send in your order at once.

Prof. J. A. Calderhead, of Curry University, Pittsburg, Pennsylvania, sent in \$3. as his subscription to the MONTHLY for 1896. We are very thankful for the material encouragement the friends of the MONTHLY are giving it.

A conference of the American Mathematical Society will convene in room 35 of Ryerson Physical Laboratory of the University of Chicago, at 10 o'clock, Thursday forenoon, December 31, 1896. It is expected that the conference will have three or four sessions and will adjourn on Friday, January 1, 1897. During the sessions of this conference some very important subjects will be discussed. Let every one interested in Mathematics attend this conference.

BOOKS AND PERIODICALS.

The Elements of Plane Geometry. By Charles A. Hobbs, A. M., Mathematical Master in the Volkmann School, Boston, Mass. 8vo. Cloth and Leather Back, 240 pages. Price, 75 cents. New York : A. Lovell & Co.

In this book the author has taken what seems to him to be a middle ground between the method of the students' following set demonstrations of a number of propositions and that of the students' producing all the argument in the course of a demonstration from original resources. There are 720 original propositions throughout the book besides many numerical exercises. The book is worthy the recognition of teachers. B. F. B.

Number and Its Algebra: A Syllabus of Lectures on the Theory of Number and Its Algebra Introductory to a Course in Algebra. By Arthur Lefevre, C. E., Instructor in Pure Mathematics, University of Texas. 8vo. Cloth, 230 pages. Boston: D. C. Heath & Co.

From only a cursory examination of this book we can say that it occupies a unique place in the literature of Mathematics. A careful reading of its contents by teachers will make the concept of numbers clear, and place their applications and the teaching of them on a solid foundation. B. F, F.

A Primer of the Calculus. By E. Sherman Gould, Member of American Society of Civil Engineers. 16mo. Boards, 92 pages. Price, 50 cents. New York: D. Van Nostrand Co.

This little work is a development of the infinitesimal Calculus as far as the first differentials of algebraic functions of one independent variable and their corresponding integrals. Its size permits it to be carried about in the coat pocket and thus the self-taught may have at his command a work which he may read and study during his leisure.
B. F. F.

Elements of the Differential Calculus. By Edgar W. Bass, Professor of Mathematics in the United States Military Academy. 12mo. Cloth, 354 pages. New York: John Wiley & Sons.

The author says: "This text-book has been prepared for the use of the cadets of the United States Military Academy who begin the subject with a knowledge of the elements of Algebra, Geometry, and Trigonometry which ranges from fair to excellent. * * * * My experience leads me to the belief that the more rigorous and comprehensive method of infinitesimals is suitable only for a treatise and not for a text-book intended for beginners."

The author has, therefore, laid the foundation of his book on the methods of limits —the most accurate and simple of all the methods of presentation. One among the many commendable features of the book is the numerous, beautiful, and accurate diagrams used to aid in establishing the various principles upon which the Calculus is based. In this respect, it will appeal most favorably to the beginner. The book is one I most heartily recommend, and it is to be hoped that the author will follow it up by an equally good work on the Integral Calculus. B. F. F.

List of Transitive Substitution Groups of Degree Twelve. By G. A. Miller, Ph. D., Göttingen, Germany. Extracted from The Quarterly Journal of Pure and Applied Mathematics, No. 111, 1896, pages 193—284.

Dr. Miller has given the subject of Substitution a great deal of study and he has written a number of articles on it. These various articles may be found in the leading Mathematical Journals of America and Europe. Those who are interested in this subject will find this article very helpful. B. F. F.

The Criterion for Two-Term Prismoidal Formulas. By Dr. George Bruce Halsted. Pamphlet, 14 pages.

This interesting and valuable paper was presented to the Texas Academy of Science at its meeting, April 5, 1896. It contains many historical references and gives a pretty full history of the development of that interesting formula. Write to Dr. Halsted for a copy.
B. F. F.

Projective Groups of Perspective Collineations in the Plane Treated Synthetically. Pamphlet, 34 pages.

A dissertation presented to the Faculty of the University of Kansas by Arnold Emch to attain the degree of Doctor of Philosophy. B. F. F.

The Outlook Illustrated Monthly Magazine, Number for October. Price, 10 cents. The Outlook Co., 13 Astor Place, New York.

This number contains a full accouut of Princeton's 150th Anniversary, by Henry Van Dyke, with pictures; The Boys' Republic, by Washington Gladden, with twelve pictures; William Morris: A Poet's Workshop, by R. F. Zueblin, with five pictures; The Founder of the Y. M. C. A., by Lord Kinnaird, with nine pictures. B. F. F.

Popular Astronomy. Edited by W. W. Payne and H. C. Wilson, Goodsell Observatory of Carlton College, Northfield, Minnesota.

The November number contains the following: The Teaching of Descriptive Astronomy; Sketch of Astronomical Work at Munich; Biography of Prof. H. A. Newton, New York Evening Post; The Theory of Probability—An Historical Sketch; The Moon; The Constitution and Function of Gases; The Twilight; The Fixed Stars; The Planets and Constellations for October; Variable Stars. B. F. F.

Prace Matematyczen-Fizyczne. Wydawane. Przez S. Dicksteina, Warsaw, Russia.

The Mathematical Gazette. Edited by F. S. Macauley, St. Paul's School, West Kensington, W. London, England. Price, 3s. per year.

The Gazette aims at satisfying a want felt by many students for a Journal of Elementary Mathematics and is especially intended to be useful to teachers. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

Errata in October Number.

Page 246, line 3, for "5^{n+1}" read 5^{n-1}.
Page 246, line 14, insert + before last term of (1).
Page 246, line 15, for "$4^{\frac{n-1}{2}}.5$" read $4^{\frac{n-2}{2}}.5$.
Page 246, line 19, insert + before last term in (2).
Page 247, line 12, for "$4626x^3$" read $4626x^5$, and for "$\times$" read +.
Page 248, line 9, complete parenthesis after numerator of next to last term.
Page 250, problem 72 should read $2\sqrt{2+\sqrt{3}}/(4+\sqrt{6}-\sqrt{2})$.
Page 251, line 7 from bottom, for "$(-x)$" read $(-a)$.
Page 252, l. 20, read $R=[F\ (C^2-4AB)+AE^2+BD^2-CD^2]/(4AB-C^2)$.
Page 252, line 2 from bottom, reverse last mark of parenthesis after F.
Page 253, line 5, for "(Em^2-2k)" read $(Em^2-2k)y$.
Page 254, line 2, second = should be +.
Page 255, line 14, for "$n=$," etc. read u.
Page 255, line 18, for "(3)" read (2).
Page 256, line 1, in denominator, for "$\sqrt[3]{m\div n}$" read $\sqrt[3]{n\div m}$.